The Classic and the Common Product

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Abstract

The common product, developed as a generalization of multiset-based similarity quantifications to real, possibly negative multiplicities, has played an important role in obtaining, through the real-valued Jaccard and coincidence indices, impressive results in several tasks including template matching, pattern recognition, neuronal networks, complex networks, and hierarchical clustering. In the present work, we address the common product from a more comprehensive mathematical perspective, including several of its interesting properties.

"*The owl awakened reflections into the calm night.*'

LdaFC

1 Introduction

The many impressive advances of science and technology have critically relied on the quantification of several static and dynamic properties of our world. In addition to allowing immediate visualizations of these measurements in terms of respective plots, measurements are also combined by using several algebraic operations including addition, subtraction, product, and division, as well as by using many other functions and operations.

While most approaches in scientific modeling and mathematics have been based on algebraic operations, other type of operations including those of Boolean algebra and set theory are also commonly employed in combined fashion. For instance, the following typical definition of the signal operation:

$$sign(x) = \begin{cases} +1 & \textit{if and only if } x > 0 \\ 0 & \textit{if and only if } x = 0 \\ -1 & \textit{if and only if } x < 0 \end{cases} \quad (1)$$

requires a combination of logic, set, relational, and algebraic concepts.

While these combinations are relatively frequent and welcomed, it is possible to develop much more integrated approaches [8, 9] in which set operations are intermingled with algebraic operations. These possibilities derive from multiset theory (e.g. [1, 2, 3, 4, 5, 6]) adaptations to take into account real, possibly negative values [7, 8, 9, 10]. One of the respectively obtained results regards the *common product* which, however, can be applied independently of the integration of set and algebraic concepts and operations.

The present work presents the mathematical context and some of the properties of the *common product* a multiset-based recently developed [7, 8, 9, 10] binary operator (in the sense of taking two arguments) that has allowed impressive performance in several areas and tasks, including template matching [11], pattern recognition [11, 12], complex networks [13], neuronal networks [14], and hierarchical clustering [15].

Observe that the common product term has been used both to designate its elementwise and functional forms, the specific meaning being disambiguated by the respective context. In the present work we focus on the *elementwise* common product.

The common product, upon which the real-valued Jaccard and coincidence indices have been based, has also shown to be a signed similarity measurement directly related to the Kronecker delta function [16]. At the same time, the common product corresponds to the intersection operation in generalized multisets.

After revising the basic mathematical context of the common product, including concepts from classic set and generalized multiset theory, we develop an analogy between the real and common products, including the presentation of several interesting properties of the latter product. The several obtained results and analogies between these two products suggest that the the common product is the counterpart of the real product respectively

to the L1 norm.

The reported properties pave the way to a number of interesting theoretical and applied possibilities in several related areas.

2 Classic Set Theory

Given three sets A, B, and C derived from a given Ω, the following properties are satisfied:

$$\Omega = \bigcup_i A_i, \quad \forall i \tag{2}$$
$$\Omega^C = \Omega - \Omega = \phi \tag{3}$$
$$\phi^C = \Omega - \phi = \Omega \tag{4}$$
$$A^C = \Omega - A \tag{5}$$
$$A \cup A^C = \Omega \tag{6}$$
$$A \cap A^C = \phi \tag{7}$$
$$A \cup \Omega = \Omega \tag{8}$$
$$A \cap \Omega = A \tag{9}$$
$$A \cup \phi = A \tag{10}$$
$$A \cap \phi = \phi \tag{11}$$
$$A \cup A = A \tag{12}$$
$$A \cap A = A \tag{13}$$
$$A \cup B = B \cup A \tag{14}$$
$$A \cap B = B \cap A \tag{15}$$
$$A \cup (B \cup C) = (A \cup B) \cup C \tag{16}$$
$$A \cap (B \cup C) = (A \cap B) \cup C \tag{17}$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{18}$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \tag{19}$$
$$(A \cup B)^C = A^C \cap B^C \text{ (De Morgan)} \tag{20}$$
$$(A \cap B)^C = A^C \cup B^C \text{ (De Morgan)} \tag{21}$$

3 Classic Multiset Concepts

By classic multiset theory we here understand multisets with non-negative integer multiplicities (e.g. [1, 2, 3, 4, 5, 6]). In particular, we understand that the multiset difference operation is clipped at zero in order to avoid negative multiplicities being respectively obtained.

These multisets have the following properties:

$$\Omega = \bigcup_i A_i, \quad \forall i \tag{22}$$
$$\Omega^C = \Omega - \Omega = \phi \tag{23}$$
$$\phi^C = \Omega - \phi = \Omega \tag{24}$$
$$A^C = \Omega - A \tag{25}$$
$$A \cup A^C = \Omega \tag{26}$$
$$* * * \quad A \cap A^C = \Omega - A \tag{27}$$
$$A \cup \Omega = \Omega \tag{28}$$
$$A \cap \Omega = A \tag{29}$$
$$A \cup \phi = A \tag{30}$$
$$A \cap \phi = \phi \tag{31}$$
$$A \cup A = A \tag{32}$$
$$A \cap A = A \tag{33}$$
$$A \cup B = B \cup A \tag{34}$$
$$A \cap B = B \cap A \tag{35}$$
$$A \cup (B \cup C) = (A \cup B) \cup C \tag{36}$$
$$A \cap (B \cup C) = (A \cap B) \cup C \tag{37}$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{38}$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \tag{39}$$

Except for the property 25, all other properties are analogous to respective counterparts in classic multiset theory.

4 Generalized Multiset Theory

The generalized multisets were developed in order to account for real, possibly negative multiplicities [7, 8, 9, 10].

Let x and y be real-valued multisets, real values, or real functions of another common variable, e.g. $x(t)$ and $y(t)$, with combined support S, $t \in S$.

A real-valued multiset x is a set of 2-tuples $[t, x(t)]$, where $x(t) \in \mathbb{R}$ is the *multiplicity* of the element t in the multiset x.

The *union* of two multisets x and y is a new multiset z given as:

$$z = x \cup y = \{\!\!\{ [t, \max\{x, y\}] \}\!\!\} \tag{40}$$

The *intersection* of two multisets x and y is a new multiset z given as:

$$z = x \cap y = [t, \min\{x, y\}] \}\!\!\} \tag{41}$$

The *addition* of two multisets x and y is a new multiset z given as:

$$z = x + y = \{\!\!\{ [t, x + y] \}\!\!\} \tag{42}$$

The *subtraction* of two multisets x and y is a new multiset z given as:

$$z = x - y = [t, x - y] \}\!\!\} \tag{43}$$

The *common union* of two multisets x and y is a new multiset z given as:

$$z = x \sqcup y = \{\!\!\{ s_{xy} \left[t, \max \left\{ s_x x, s_y y \right\} \right] \}\!\!\} \tag{44}$$

where $s_x = sign(x)$, $s_y = sign(y)$, and $s_{xy} = s_x s_y$.

The *common intersection* of two multisets x and y is a new multiset z given as:

$$z = x \sqcap y = \{\!\!\{ s_{xy} \left[t, \min \left\{ s_x x, s_y y \right\} \right] \}\!\!\} \tag{45}$$

The *absolute union* of two multisets x and y is a new multiset z given as:

$$z = x \tilde{\sqcup} y = \{\!\!\{ \left[t, \max \left\{ s_x x, s_y y \right\} \right] \}\!\!\} \tag{46}$$

The *absolute intersection* of two multisets x and y is a new multiset z given as:

$$z = x \tilde{\sqcap} y = \{\!\!\{ \left[t, \min \left\{ s_x x, s_y y \right\} \right] \}\!\!\} \tag{47}$$

We have the following, non-exhaustive, properties of generalized multisets:

$$\Omega_+ = \bigcup_i x, \quad \forall x \tag{48}$$
$$\Omega_- = \bigcap_i x, \quad \forall x \tag{49}$$
$$\Omega_+^C = \phi - \Omega_+ = -\Omega_+ \tag{50}$$
$$\Omega_-^C = \phi - \Omega_- = -\Omega_- \tag{51}$$
$$\phi^C = \phi - \phi = \phi \tag{52}$$
$$x^C = \phi - x = -x \tag{53}$$
$$x \cup x^C = x \cup (-x) = |x| \tag{54}$$
$$x \cap x^C = x \cap (-x) = -|x| \tag{55}$$
$$x \cup \Omega_+ = \Omega_+ \tag{56}$$
$$x \cup \Omega_- = x \tag{57}$$
$$x \cap \Omega_+ = x \tag{58}$$
$$x \cap \Omega_- = \Omega_- \tag{59}$$
$$x \cup \phi = x \tag{60}$$
$$x \sqcap \phi = \phi \tag{61}$$
$$x \cup x = x \tag{62}$$
$$x \cap x = x \tag{63}$$
$$x \cup y = y \cup x \tag{64}$$
$$x \cap y = y \cap x \tag{65}$$
$$x \cup (y \cup z) = (x \cup y) \cup z \tag{66}$$
$$x \cap (y \cup z) = (x \cap y) \cup z \tag{67}$$
$$x \cap (y \cup z) = (x \cap y) \cup (x \cap z) \tag{68}$$
$$x \cup (y \cap z) = (x \cup y) \cap (x \cup z) \tag{69}$$
$$(x \cup y)^C = x^C \cap y^C \text{ (De Morgan)} \tag{70}$$
$$(x \cap y)^C = x^C \cup y^C \text{ (De Morgan)} \tag{71}$$

where:

$$A \sqcap B = s_{A_i B_i} \min \left\{ s_{A_i} A_i, s_{B_i} B_i \right\} \in \mathbb{R} \tag{72}$$

with $s_{A_i} = sign(A_i)$, $s_{A_i} + sign(A_i)$, and $S_{A_i B_i} = s_{A_i} s_{B_i}$

In the case of all possible multisets, we have $\Omega_+ = +\infty$ and $\Omega_- = -\infty$. In addition, observe that the complement operation is take respectively to the empty set, and not the universe set as in classic set theory.

5 The Real Product

Given any two real values x and y, their product corresponds to the binary operation:

$$p(x, y) = xy \in \mathbb{R} \tag{73}$$

Given any two real functions $x(t)$ and $y(t)$, their product corresponds to the binary operation:

$$p(x(t), y(t)) = x(t) y(t) \in \mathbb{R} \tag{74}$$

For simplicity's sake, we henceforth use x and y to represent both real values or real functions.

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The properties of the real product binary operator include:

$$xy = yx \tag{75}$$
$$0x = 0 \tag{76}$$
$$1x = x \tag{77}$$
$$x(yz) = (xy)z \tag{78}$$
$$x(y + z) = xy + xz \tag{79}$$
$$xx \geq 0, \forall x \in \mathbb{R} \tag{80}$$
$$xx = x^2 \tag{81}$$
$$\sqrt{x^2} = |x| \tag{82}$$
$$(xy)^a = x^a \; y^a \tag{83}$$
$$sign(xy) = sign(x) \; sign(y) \tag{84}$$
$$sign(x) \; x = |x| \tag{85}$$
$$\log(xy) = \log(x) + \log(y) \tag{86}$$
$$\frac{dxy}{dt} = x \frac{dy}{dt} + y \frac{dx}{dt} \tag{87}$$
$$(ax)y = a(xy) \tag{88}$$
$$(ax)(ay) = a^2 (xy) \tag{89}$$

6 The Common Product

Given any two real values x and y, their *common product* corresponds to the binary operation:

$$p(x, y) = s_{xy} \min \left\{ s_x x, s_y y \right\} = x \sqcap y \in \mathbb{R} \tag{90}$$

It has been shown that this product is a similarity measurement related to the Kronecker delta function [16].

Let $g(x)$ be a *non-decreasing monotonic function*, in the sense that $x \geq y \implies g(x) \geq g(y)$, also being odd, i.e. $g(-x) = -g(x)$.

Then, we can prove that $g(x \sqcap y) = g(x) \sqcap g(y)$. This can be done in elementwise manner (i.e. x and y are considered real values) considering the following four situations.

Situation 1: $s_x = s_y = +1$. Then, we have that:

$$x \sqcap y = s_{xy} \min\{s_x x, s_y y\} = \min\{x, y\} \tag{91}$$
$$g(x \sqcap y) = g(\min\{x, y\}) \tag{92}$$
$$g(x) \sqcap g(y) = \min\{g(x), g(y)\} \tag{93}$$

If $x < y$, we have:

$$g(x \sqcap y) = g(\min\{x, y\}) = g(x) \tag{94}$$
$$g(x) \sqcap g(y) = \min\{g(x), g(y)\} = g(x) \tag{95}$$

If $x \geq y$, we have:

$$g(x \sqcap y) = g(\min\{x, y\}) = g(y) \tag{96}$$
$$g(x) \sqcap g(y) = \min\{g(x), g(y)\} = g(y) \tag{97}$$

Therefore, it follows that, for $s_x = s_y = +1$ we have $g(x \sqcap y) = g(x) \sqcap g(y)$.

Situation 2: $s_x = s_y = -1$, implying:

$$x \sqcap y = s_{xy} \min\{s_x x, s_y y\} = \min\{|x|, |y|\} \tag{98}$$

which is analogous to Situation 1.

Situation 3: $s_x = 1, s_y = -1$. Then, it follows that:

$$x \sqcap y = s_{xy} \min\{s_x x, s_y y\} = -\min\{|x|, |y|\} \tag{99}$$

Situation 4: $s_x = -1, s_y = 1$, yielding:

$$x \sqcap y = s_{xy} \min\{s_x x, s_y y\} = -\min\{|x|, |y|\} \tag{100}$$

Given that Situations 3 and 4 are analogous, it is enough to prove the latter. We start with:

$$g(x \sqcap y) = g(-\min\{|x|, |y|\}) \tag{101}$$
$$g(x) \sqcap g(y) = -\min\{|g(x)|, |g(y)|\} \tag{102}$$

If $|x| < |y|$, we have:

$$g(x \sqcap y) = g(-\min\{|x|, |y|\}) = g(-|x|) \tag{103}$$
$$g(x) \sqcap g(y) = -\min\{|g(x)|, |g(y)|\} = = -|g(x)| = g(-|x|) \tag{104}$$

If $|x| \geq |y|$, we have:

$$g(x \sqcap y) = g(-\min\{|x|, |y|\}) = g(-|y|) \tag{105}$$
$$g(x) \sqcap g(y) = -\min\{|g(x)|, |g(y)|\} = -|g(y)| \tag{106}$$

QED.

Additional analogous properties can be obtained for non-increasing and even functions.

The properties of the common product binary operator include:

$$x \sqcap y = y \sqcap x \tag{107}$$
$$\phi \sqcap x = \phi \tag{108}$$
$$x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z \tag{109}$$
$$x \sqcap (y \sqcup z) = x \sqcap y \sqcup x \sqcap z \tag{110}$$
$$x \sqcap x = |x| \tag{111}$$
$$\sqrt{x \sqcap x} = \sqrt{|x|} \tag{112}$$
$$(x \sqcap y)^a = x^a \sqcap y^a, \text{ with } a \text{ integer and odd} \tag{113}$$
$$sign(x \sqcap y) = sign(x)\ sign(y) \tag{114}$$
$$sign(x) \sqcap x = |x| \tag{115}$$
$$(ax) \sqcap (ay) = a(x \sqcap y) \tag{116}$$

Equations 111 and 112 indicates that the common product is to the L1 norm as the real product is to the L2 norm.

7 Concluding Remarks

In this work, we have provided a relatively comprehensive mathematical context to the common product, a binary operation that has allowed several impressive results in several areas.

After presenting the basic concepts and properties from classic set and generalized multiset theories, we developed a comparative study of the real and common product, including the derivation of several properties of the latter. The common product is shown to correspond to the intersection operation in generilized multisetd.

It has also been verified that the common product presents several properties that are analogous to those of the real product, including being commutative, having null element, and being associative and distributive. Of particular interest is the fact that the common product is to the L1 norm as the real product is to the L2 norm. In addition, it has been shown that the common product result the same result when composed with monotonically non-decreasing odd functions. A composed derivative property has also been identified. As expected, the common product is not a bilinear operator, though being distributed with respect to the scalar product.

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